Intermediate Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the Intermediate Mathematical Olympiad and similar competitions).

Enquiries about the Intermediate Mathematical Challenge should be sent to:

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	B	A	D	B	D	C	D	B	A	E	D	A	B	C	D	B	E	E	D	A	C	D	C	A

1. What is the value of $\frac{2}{5} + \frac{2}{50} + \frac{2}{500}$?

A 0.111 B 0.222 C 0.333 D 0.444 E 0.555

SOLUTION **D**

We have $\frac{2}{5} = 0.4$. Multiplying by $\frac{1}{10}$ is the same as dividing by 10, and therefore has the effect of moving the decimal point one place to the left. Therefore $\frac{2}{50} = \frac{1}{10} \times \frac{2}{5} = 0.04$ and $\frac{2}{500} = \frac{1}{10} \times \frac{2}{50} = 0.004$. Hence $\frac{2}{5} + \frac{2}{50} + \frac{2}{500} = 0.4 + 0.04 + 0.004 = 0.444$.

FOR INVESTIGATION

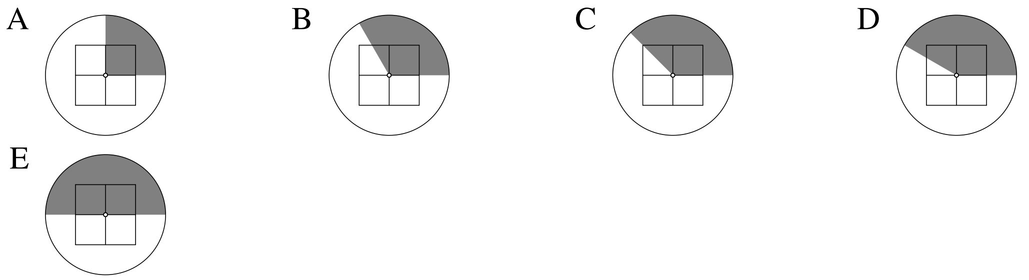
1.1 Write as a decimal the answer to the sum

$$\frac{2}{5} + \frac{2}{50} + \frac{2}{500} + \frac{2}{5000} + \frac{2}{50\,000} + \frac{2}{500\,000} + \frac{2}{5\,000\,000}.$$

1.2 What is the decimal representation of the fraction $\frac{4}{9}$? How does this compare with your answer to 1.1? What conclusion do you draw?

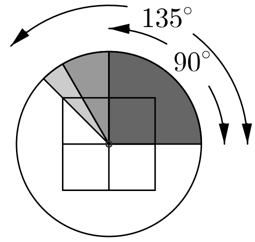
2. Each of the diagrams below shows a circle and four small squares. In each case, the centre of the circle is the point where all four squares meet.

In one of the diagrams, exactly one third of the circle is shaded. Which one?



SOLUTION **B**

The straight edges of the shaded sector which occupies one third of the area of the circle will make an angle equal to one third of a complete revolution, that is, an angle of $\frac{1}{3} \times 360° = 120°$.



In diagram A the straight edges of the shaded sector make an angle of 90°. In diagram C these edges make an angle equal to $90°+45° = 135°$. The angles in diagrams D and E are even larger. So in none of these cases is exactly one third of the circle shaded.

This leaves diagram B where the angle is between 90° and 135°. Therefore, this is the diagram where the angle is 120°. Hence it is the diagram where the shaded area occupies one third of the circle.

3. How many squares have 7 as their units digit?

A 0 B 1 C 2 D 3 E 4

SOLUTION **A**

The units digit of the square of an integer n is the same as that of the square of the units digit of n. [You are asked to prove this in Problem 3.1.] For example, the units digit of 237^2 is the same as that of 7^2, namely 9.

Therefore, to find the possible units digits of squares, we need only consider the squares of the one-digit numbers. We have $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$ and $9^2 = 81$. This shows that the units digit of a square can only be 0, 1, 4, 5, 6 or 9. In particular, there are are no squares which have 7 as their units digit.

FOR INVESTIGATION

3.1 Prove that the units digit of the square of an integer n is the same as the units digit of the square of the units digit of n. [Hint: Show that if n, m and u are positive integers with $n = 10m + u$, then n^2 and u^2 have the same remainder when they are divided by 10.]

3.2 What are the possible remainders when $x^2 + 3xy + y^2$, is divided by 10, where x and y are positive integers? Deduce that the equation $x^2 + 3xy + y^2 = 222\,222$ has no positive integer solutions.

4. Which of the following is *not* the sum of two primes?

A 5 B 7 C 9 D 11 E 13

SOLUTION **D**

The sum of two odd numbers is even. Therefore, no odd number is the sum of two odd primes. The only even prime is 2. It follows that if any of the odd numbers 5, 7, 9, 11 and 13 is the sum of two primes, it is the sum of an odd prime and 2.

We see that $5 = 3 + 2$, $7 = 5 + 2$, $9 = 7 + 2$, $11 = 9 + 2$ and $13 = 11 + 2$.

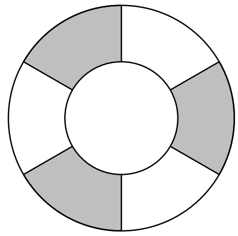
The numbers 3, 5, 7 and 11 are all primes, so each of 5, 7, 9 and 13 is the sum of two primes. However, 9 is not a prime and therefore 11 is not the sum of two primes.

NOTE

As we show above, the only odd prime numbers that are the sum of two primes are those of the form $p + 2$, where p is an odd prime. In this case the numbers p, $p + 2$ form what is called a *prime pair*, that is, they are consecutive odd numbers that are both prime. It is an unsolved problem as to whether there are infinitely many prime pairs.

According to the *Goldbach Conjecture* every even number greater than 2 is the sum of two primes. This has not been proved but, at the time of writing, it has been verified for all even numbers up to 4×10^{18}. Some partial results are known. For example, Chen's Theorem (proved by Chen Jingrun in 1973) says that all but a finite number of even numbers can be expressed as the sum of a prime and a number which is either prime or the product of two primes.

5. The diagram shows two circles with the same centre. The radius of the outer circle is twice the radius of the inner circle. The region between the inner circle and the outer circle is divided into six equal segments as shown.



What fraction of the area of the outer circle is shaded?

A $\frac{3}{7}$ B $\frac{3}{8}$ C $\frac{3}{9}$ D $\frac{3}{10}$ E $\frac{3}{11}$

SOLUTION **B**

We suppose that the radius of the inner circle is r. It follows that the radius of the outer circle is $2r$. Therefore the inner circle has area πr^2 and the outer circle has area $\pi(2r)^2$, that is, $4\pi r^2$.

Hence, the area of the region between the inner and outer circles is $4\pi r^2 - \pi r^2 = 3\pi r^2$. Half of this region is shaded. Therefore the shaded area is $\frac{3}{2}\pi r^2$.

It follows that the fraction of the area of the outer circle that is shaded is given by

$$\frac{\frac{3}{2}\pi r^2}{4\pi r^2} = \frac{3}{8}.$$

6. The angles of a quadrilateral are in the ratio 3 : 4 : 5 : 6.

What is the difference between the largest angle and the smallest angle?

A 30° B 40° C 50° D 60° E 70°

SOLUTION **D**

Because the angles are in the ratio 3 : 4 : 5 : 6, and $3 + 4 + 5 + 6 = 18$, the largest angle makes up $\frac{6}{18}$, that is, $\frac{1}{3}$ of the sum of the angles of the quadrilateral. Similarly, the smallest angle makes up $\frac{3}{18}$, that is, $\frac{1}{6}$ of the sum of the angles of the quadrilateral.

It follows that the difference between the largest and the smallest angle is $\frac{1}{3} - \frac{1}{6}$, that is, $\frac{1}{6}$ of the sum of the angles of the quadrilateral.

The sum of the angles of a quadrilateral is 360°. It follows that the difference between the largest angle and the smallest angle of the quadrilateral is

$$\frac{1}{6} \times 360^\circ = 60^\circ.$$

FOR INVESTIGATION

6.1 What are the four angles of a quadrilateral whose angles are in the ratio 3 : 4 : 5 : 6?

6.2 The angles of a pentagon are in the ratio 3 : 4 : 5 : 6 : 7. What is the difference between the largest angle and the smallest angle?

6.3 Prove that the sum of the angles of a quadrilateral is 360°.

7. Four different positive integers are to be chosen so that they have a mean of 2017.

What is the smallest possible range of the chosen integers?

A 2 B 3 C 4 D 5 E 6

SOLUTION **C**

We recall that the *range* of a set of numbers is the difference between the largest and smallest numbers in the set.

The smallest range of a set of four different positive integers is 3. This occurs when the integers are consecutive, and only in this case. We first show that we cannot have four consecutive integers with a mean of 2017.

Assume, to the contrary, that n is a positive integer such that the consecutive integers n, $n+1$, $n+2$ and $n+3$ have 2017 as their mean.

Then
$$\tfrac{1}{4}(n + (n + 1) + (n + 2) + (n + 3)) = 2017,$$
and hence
$$n + (n + 1) + (n + 2) + (n + 3) = 4 \times 2017.$$
It follows that
$$4n + 6 = 4 \times 2017,$$
and hence
$$4n = 4 \times 2017 - 6.$$
Therefore
$$n = 2017 - \tfrac{3}{2},$$
contradicting our assumption that n is an integer. This contradiction shows that there do not exist four consecutive integers whose mean is 2017. So the smallest possible range is not 3.

On the other hand, we see that the four integers 2015, 2016, 2018 and 2019 have mean 2017, because

$$\begin{aligned} 2015 + 2016 + 2018 + 2019 &= (2015 + 2019) + (2016 + 2018) \\ &= 2 \times 2017 + 2 \times 2017 \\ &= 4 \times 2017. \end{aligned}$$

The range of these numbers is $2019 - 2015$, that is, 4. We deduce that the smallest possible range is 4.

FOR INVESTIGATION

7.1 Five different positive integers are to be chosen so that their mean is 2017. What is the smallest possible range of the chosen integers?

7.2 Note that there is nothing special in this context about the number 2017. Show how the argument above may be adapted to show that the mean of four consecutive integers is never an integer.

7.3 Investigate for which positive integers k it is possible to find k consecutive integers whose mean is an integer.

8. Which of the following numbers is the largest?

A 1.3542 B $1.354\dot{2}$ C $1.35\dot{4}\dot{2}$ D $1.3\dot{5}4\dot{2}$ E $1.\dot{3}54\dot{2}$

Solution **D**

To compare the numbers, we give them to 8 decimal places, as follows.

$$\begin{aligned}
1.3542 &\approx 1.3542\mathbf{0}000\\
1.354\dot{2} &\approx 1.3542\mathbf{2}222\\
1.35\dot{4}\dot{2} &\approx 1.3542\mathbf{4}242\\
1.3\dot{5}4\dot{2} &\approx 1.3542\mathbf{5}425\\
1.\dot{3}54\dot{2} &\approx 1.3542\mathbf{3}542
\end{aligned}$$

These numbers agree in their first four decimal places. The number $1.3\dot{5}4\dot{2}$ has a larger digit in the fifth decimal place than the others. Therefore $1.3\dot{5}4\dot{2}$ is the largest of the listed numbers.

9. The number 'tu' is the two-digit number with units digit u and tens digit t. The digits a and b are distinct, and non-zero.

What is the largest possible value of 'ab' − 'ba'?

A 81 B 72 C 63 D 54 E 45

Solution **B**

The place-value notation means that 'ab' $= 10a + b$ and 'ba' $= 10b + a$. Therefore

$$\begin{aligned}
\text{‘}ab\text{’} - \text{‘}ba\text{’} &= (10a + b) - (10b + a)\\
&= (10a - a) + (b - 10b)\\
&= 9a - 9b\\
&= 9(a - b).
\end{aligned}$$

The largest possible value of $9(a - b)$ occurs when $a - b$ has its largest possible value. Because a and b are different non-zero digits, the largest value of $a - b$ is 8 when $a = 9$ and $b = 1$. In this case 'ab' − 'ba' $= 91 - 19 = 72$.

Therefore the largest possible value of 'ab' − 'ba' $= 72$.

Note

In Problems 9.1 and 9.2 we use the notation 'htu' for the three-digit number with units digit u, tens digit t and hundreds digit h.

For investigation

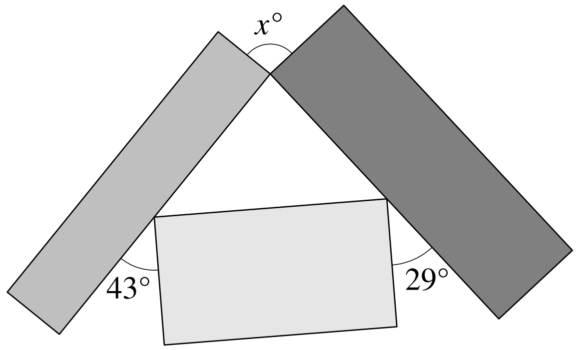
9.1 The digits a, b and c are distinct and non-zero. What is the largest possible value of 'abc' − 'cba'?

9.2 Let a, b and c be digits with $a > c$. Suppose that 'abc' − 'cba' = 'def', where d, e and f are digits. Investigate the possible values of 'def' + 'fed'.

10. The diagram shows three rectangles.

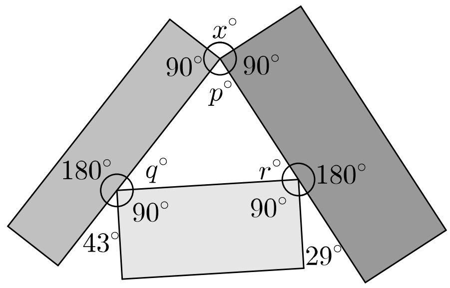
What is the value of x?

A 108 B 104 C 100 D 96 E 92



Solution **A**

We let the angles in the triangle that is formed by the rectangles be $p°$, $q°$ and $r°$, as shown in the figure.



The angles of a rectangle are 90°, as shown. The angles on the line at the points where the bottom rectangle meets the other two are each 180°, as shown.

The total of the angles at a point is 360°. Therefore, from the angles at the points where the rectangles meet, we have the following three equations.

$$\begin{aligned} p + 90 + 90 + x &= 360, \\ q + 180 + 90 + 43 &= 360, \\ \text{and} \qquad r + 180 + 90 + 29 &= 360. \end{aligned}$$

It follows that

$$\begin{aligned} p &= 180 - x, \\ q &= 47, \\ \text{and} \qquad r &= 61. \end{aligned}$$

Therefore, because the angles in a triangle total 180°,

$$(180 - x) + 47 + 61 = 180,$$

from which it follows that

$$180 - x + 108 = 180,$$

and hence

$$x = 108.$$

For investigation

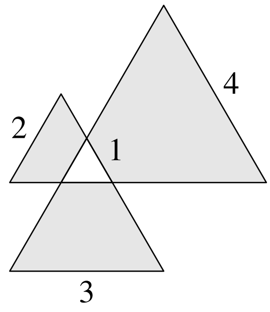
10.1 What is the value of x in the case where the angles between the lower rectangle and the other two rectangles are 37° and 31°?

10.2 Let the angles between the lower rectangle and the other rectangles be $a°$ and $b°$. Find a formula for x in terms of a and b.

11. The diagram shows four equilateral triangles with sides of length 1, 2, 3 and 4. The area of the shaded region is equal to n times the area of the unshaded triangle of side-length 1.

What is the value of n?

A 8 B 11 C 18 D 23 E 26



Solution **E**

We suppose that the equilateral triangle with sides of length 1 has area a. Because the areas of similar figures are proportional to the squares of their side lengths, it follows that the equilateral triangles with side lengths 2, 3 and 4, have areas $4a$, $9a$ and $16a$, respectively.

It follows that the areas of the parts of these triangles that are shaded are $4a - a$, $9a - a$ and $16a - a$, that is, $3a$, $8a$ and $15a$, respectively.

Therefore, the total shaded area is $3a + 8a + 15a = 26a$. Hence $n = 26$.

For investigation

11.1 Note that we solved this problem without the need to work out the area of an equilateral triangle with side length 1. What is the area of this triangle?

11.2 Show that if two similar triangles have corresponding side lengths in the ratio $k : m$, then their areas are in the ratio $k^2 : m^2$.

12. The combined age of Alice and Bob is 39. The combined age of Bob and Clare is 40. The combined age of Clare and Dan is 38. The combined age of Dan and Eve is 44. The total of all five ages is 105.

Which of the five is the youngest?

A Alice B Bob C Clare D Dan E Eve

Solution **D**

Eve's age is the total of all their ages, less the ages of Alice, Bob, Clare and Dan. Therefore we can find Eve's age by subtracting from the total of all five ages both the combined total of the ages of Alice and Bob, and the combined total of the ages of Clare and Dan.

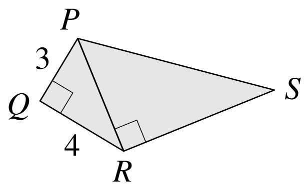
Therefore Eve's age is

$$105 - 39 - 38 = 28.$$

Since the combined age of Dan and Eve is 44, we deduce that Dan is 16. Since the combined age of Clare and Dan is 38, it follows that Clare is 22. Since the combined age of Bob and Clare is 40, Bob is 18. Finally, since the combined age of Alice and Bob is 39, Alice is 21.

Hence the youngest is Dan.

13. The diagram shows a quadrilateral $PQRS$ made from two similar right-angled triangles, PQR and PRS. The length PQ is 3, the length QR is 4 and $\angle PRQ = \angle PSR$.



What is the perimeter of $PQRS$?

A 22 B $22\frac{5}{6}$ C 27 D 32 E $45\frac{1}{3}$

Solution **A**

Applying Pythagoras' Theorem to the right-angled triangle PQR gives $RP^2 = PQ^2 + QR^2 = 3^2 + 4^2 = 9 + 16 = 25$. Therefore RP has length 5.

The corresponding sides PQ and PR of the similar triangles PQR and PRS have lengths 3 and 5. It follows that the ratio of the perimeter of PQR to that of PRS is 3 : 5.

The perimeter of the triangle PQR is $PQ + QR + RP = 3 + 4 + 5 = 12$. Let the perimeter of PRS be p. Then $12 : p = 3 : 5$. Hence

$$\frac{12}{p} = \frac{3}{5}.$$

Hence

$$p = \frac{5}{3} \times 12 = \frac{60}{3} = 20.$$

The perimeter of $PQRS$ is $PQ + QR + RS + SP$. This is equal to $((PQ + QR + RP) - RP) + ((RS + SP + RP) - RP$. Thus the perimeter equals the sum of the perimeters of PQR and PRS less twice the length of RP. We deduce that the perimeter of $PQRS$ is $12 + 20 - 2 \times 5 = 32 - 10 = 22$.

14. For what value of x is 64^x equal to 512^5?

A 6 B 7.5 C 8 D 16 E 40

Solution **B**

We note first that $64 = 2^6$ and $512 = 2^9$. Therefore $64^x = (2^6)^x = 2^{6x}$, and $512^5 = (2^9)^5 = 2^{9\times5} = 2^{45}$.

It follows that the equation $64^x = 512^5$ is equivalent to the equation $2^{6x} = 2^{45}$. We deduce from this that $6x = 45$. Hence

$$x = \frac{45}{6} = \frac{15}{2} = 7.5.$$

For investigation

14.1 Find the value of x in each of the following equations.

(a) $32^x = 1024^3$.

(b) $16 \times 32^x = 128^x$.

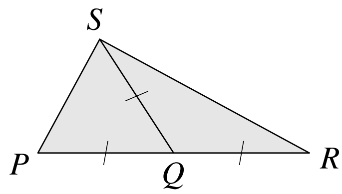
(c) $64^x = 2$.

(d) $9^x = 243$.

15. In the diagram shown, $PQ = SQ = QR$ and $\angle SPQ = 2 \times \angle RSQ$.

What is the size of angle QRS?

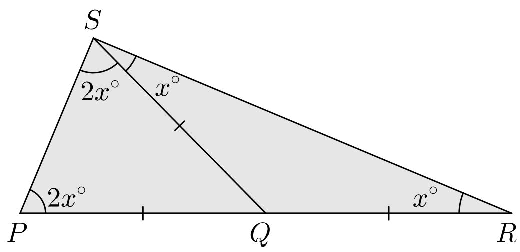
A $20°$ B $25°$ C $30°$ D $35°$ E $40°$



Solution **C**

Method 1

We let $\angle QRS = x°$. Because $SQ = QR$, the triangle SQR is isosceles. Therefore $\angle RSQ = \angle QRS = x°$.

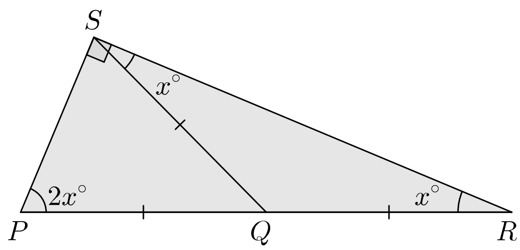


Because $\angle SPQ = 2 \times \angle RSQ$, we deduce that $\angle SPQ = 2x°$. Since $PQ = SQ$, the triangle SPQ is isosceles . Therefore $\angle PSQ = \angle SPQ = 2x°$. It follows that $\angle RSP = 2x° + x° = 3x°$.

We now apply the fact that the angles of a triangle have sum 180° to the triangle PRS. This gives $x + 2x + 3x = 180$. Hence $6x = 180$. Therefore $x = 30$. It follows that $\angle QRS$ is $30°$.

Method 2

We let $\angle QRS = x°$. Because $SQ = QR$, the triangle SQR is isosceles. Therefore $\angle RSQ = \angle QRS = x°$. Because $\angle SPQ = 2 \times \angle RSQ$, we deduce that $\angle SPQ = 2x°$.



Because $PQ = SQ = QR$, the point S lies on the circle with centre Q which goes through P and R. It follows that $\angle RSP$ is the angle in a semicircle and hence is $90°$.

We now apply the fact that the angles of a triangle have sum 180° to the triangle PRS. This gives $x + 2x + 90 = 180$. Hence $3x = 90$. Therefore $x = 30$. It follows that $\angle QRS$ is $30°$.

For investigation

15.1 Suppose that, as in the question, $PQ = SQ = QR$, but $\angle SPQ = 3 \times \angle RSQ$. Find the size of angle QRS.

15.2 Suppose that, as in the question, $PQ = SQ = QR$, but $\angle SPQ = k \times \angle RSQ$, where k is a positive number. Find the size of angle QRS in terms of k.

15.3 Our second method used the theorem that *the angle in a semicircle is a right angle*. Give a proof of this theorem.

16. The product of two positive integers is equal to twice their sum. This product is also equal to six times the difference between the two integers.

What is the sum of these two integers?

A 6 B 7 C 8 D 9 E 10

Solution **D**

We let the two positive integers be m and n, with $m \geq n$.

From the information in the question we deduce that

$$mn = 2(m+n)$$

and

$$mn = 6(m-n).$$

It follows that

$$6(m-n) = 2(m+n).$$

Hence

$$6m - 6n = 2m + 2n.$$

It follows that

$$4m = 8n,$$

and therefore

$$m = 2n.$$

Substituting $2n$ for m in the equation $mn = 2(m+n)$, gives

$$2n^2 = 6n.$$

Because $n \neq 0$, we can divide both sides of this last equation by $2n$ to deduce that

$$n = 3.$$

Therefore $m = 6$. Hence

$$\begin{aligned} m + n &= 6 + 3 \\ &= 9. \end{aligned}$$

Therefore the sum of the two integers is 9.

For investigation

16.1 The product of two positive numbers m and n is equal to three times their sum and nine times their difference. Find the values of m and n.

16.2 Let k and l be positive numbers with $l > k$. The product of the positive numbers m and n is k times their sum and l times their difference.

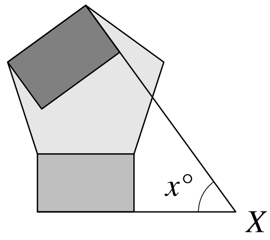
(a) Find formulas for m and n in terms of k and l.

(b) Check that if you substitute $k = 3$ and $l = 9$ in the formulas that you found in (a) above, you obtain the same values for m and n that you found in 16.1.

17. The diagram shows two rectangles and a regular pentagon. One side of each rectangle has been extended to meet at X.

What is the value of x?

A 52 B 54 C 56 D 58 E 60



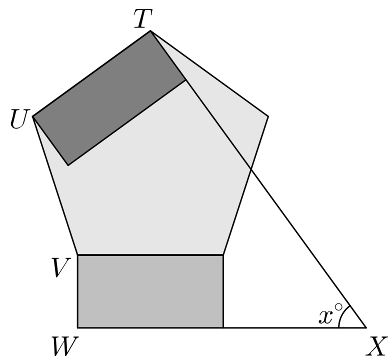
SOLUTION **B**

NOTE

The value of x may be found in several ways. We describe three of these here. There are others.

METHOD 1

We use the fact that the sum of the angles of a pentagon is 540°. Hence, each interior angle of a regular pentagon is 108°. We also know that the interior angles of the rectangles are each 90°.

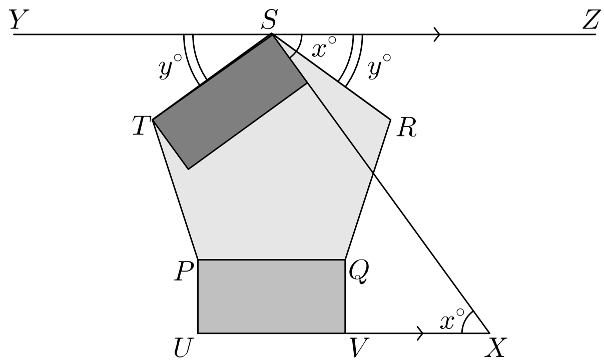


We now consider the pentagon $TUVWX$, as shown in the figure. The interior angles of this polygon at T and W are each 90°. The interior angle at U is the interior angle of the regular polygon, namely 108°. The interior angle at V is the sum of an interior angle of the regular pentagon and a right angle, that is, $108° + 90°$.

Therefore $90 + 108 + (108 + 90) + 90 + x = 540$. Hence, $x + 486 = 540$. Therefore $x = 540 - 486 = 54$.

METHOD 2

We label the vertices of the pentagon and one of the rectangles as shown in the figure. Let YZ be a line through S that is parallel to UV. It follows that the alternate angles, $\angle SXU$ and $\angle XSZ$ are equal. Hence $\angle XSZ = x°$.



By the symmetry of the pentagon about the line through S perpendicular to YZ, we have $\angle YST = \angle RSZ$. Let $\angle YST = \angle RSZ = y°$.

The angles on the line at the point S have sum 180°. Because it is the angle of a regular pentagon. $\angle TSR = 108°$. Because it is the angle of a rectangle $\angle TSX = 90°$. Therefore

$$2y + 108 = 180$$

and

$$x + y + 90 = 180.$$

From the first of these equations it follows that

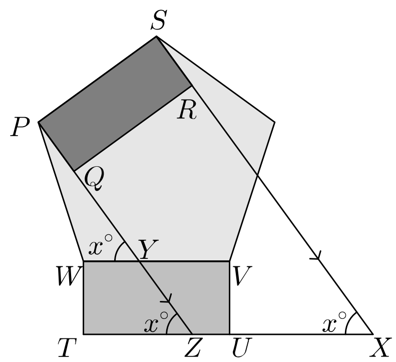
$$\begin{aligned} y &= \tfrac{1}{2}(180 - 108) \\ &= \tfrac{1}{2}(72) = 36. \end{aligned}$$

Therefore from the second equation we deduce that

$$x = 180 - 36 - 90 = 54.$$

METHOD 3

Let the rectangles be $PQRS$ and $TUVW$, labelled as shown in the figure.



Let the edge PQ when extended meet WV at Y and TU at Z.

The lines SX and PZ are extensions of opposite edges of the rectangle $PQRS$ and hence they are parallel. Therefore the corresponding angles $\angle SXU$ and $\angle PZT$ are equal. Therefore $\angle PZT = x°$.

Because the edges WV and TU of the rectangle $TUVW$ are parallel, the corresponding angles $\angle PYW$ and $\angle PZT$ are equal. Therefore $\angle PYW = x°$.

We now consider the other two angles of the triangle PYW. First, $\angle YWP = 108°$ since it is an angle of the regular pentagon. Next, $\angle WPY$ is the angle $\angle WPS$ of the regular pentagon minus the angle $\angle QPS$ of the rectangle $PQRS$. Therefore $\angle WPY = 108° - 90° = 18°$.

Because the angles of the triangle PTW have sum 180°,

$$x + 108 + 18 = 180$$

and therefore

$$\begin{aligned} x &= 180 - 108 - 18 \\ &= 54. \end{aligned}$$

FOR INVESTIGATION

17.1 We have used the fact that the sum of the interior angles of a pentagon is 540° and hence that each angle of a regular pentagon is 108°. Show that this is correct.

17.2 In Method 2, we took it for granted that *by symmetry* $\angle YST = \angle RSZ$. Show that this is correct.

18. A water tank is $\frac{5}{6}$ full. When 30 litres of water are removed from the tank, the tank is $\frac{4}{5}$ full.

How much water does the tank hold when full?

A 180 litres B 360 litres C 540 litres D 720 litres E 900 litres

SOLUTION **E**

The 30 litres of water removed from the tank is the difference between $\frac{5}{6}$ths and $\frac{4}{5}$ths of the capacity of the tank. Now

$$\frac{5}{6} - \frac{4}{5} = \frac{5 \times 5 - 4 \times 6}{5 \times 6} = \frac{25 - 24}{30} = \frac{1}{30}.$$

We therefore see that 30 litres amounts to $\frac{1}{30}$th of the capacity of the tank. It follows that when the tank is full the number of litres that it holds is $30 \times 30 = 900$.

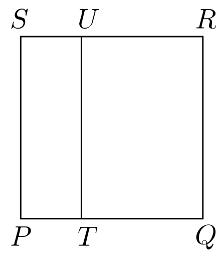
19. $PQRS$ is a square. Point T lies on PQ so that $PT : TQ = 1 : 2$. Point U lies on SR so that $SU : UR = 1 : 2$. The perimeter of $PTUS$ is 40 cm.

What is the area of $PTUS$?

A $40\,\text{cm}^2$ B $45\,\text{cm}^2$ C $48\,\text{cm}^2$ D $60\,\text{cm}^2$ E $75\,\text{cm}^2$

SOLUTION **E**

Because $PQRS$ is a square, $SR = PQ$ and $PS = PQ$. Because the ratio $PT : TQ$ is $1 : 2$, we deduce that $PT = \frac{1}{3}PQ$. Similarly, we have $SU = \frac{1}{3}SR = \frac{1}{3}PQ = PT$.



The lines PT and and SU are parts of opposite sides of the square $PQRS$. Therefore they are parallel. Because they are also of equal length, it follows that $PTUS$ is a parallelogram. Therefore $TU = PS = PQ$.

It follows that the perimeter of $PTUS$ is $PT + TU + US + SP = \frac{1}{3}PQ + PQ + \frac{1}{3}PQ + PQ = \frac{8}{3}PQ$. Therefore $\frac{8}{3}PQ = 40\,\text{cm}$. Hence $PQ = \frac{3}{8} \times 40\,\text{cm} = 15\,\text{cm}$. Therefore $PT = \frac{1}{3} \times 15\,\text{cm} = 5\,\text{cm}$.

We also note that, because $\angle SPT = \angle PSU = 90°$, we can deduce that $PTUS$ is a rectangle.

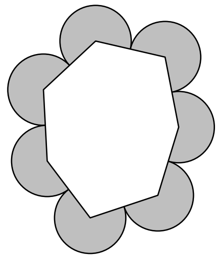
It follows that the area of $PTUS$ is given by

$$PT \times TU = 5\,\text{cm} \times 15\,\text{cm} = 75\,\text{cm}^2.$$

FOR INVESTIGATION

19.1 In the above solution we said that because the lines PT and SU are parallel and of equal length, it follows that $PTUS$ is a parallelogram. Prove that this is the case by showing that SP is parallel to UT.

20. The diagram shows seven circular arcs and a heptagon with equal sides but unequal angles. The sides of the heptagon have length 4. The centre of each arc is a vertex of the heptagon, and the ends of the arc are the midpoints of the two adjacent sides.



What is the total shaded area?

A 12π B 14π C 16π D 18π E 20π

Solution **D**

The shaded area is made up of seven sectors of circles each of radius 2.

The area of a circle with radius 2 is $\pi \times 2^2 = 4\pi$.

The area of a sector of a circle is directly proportional to the angle subtended by the sector at the centre of the circle. A full circle subtends an angle of 360° at its centre. Therefore, if the total of the angles subtended by the shaded sectors shown in the figure is $x \times 360°$, the total shaded area is $x \times 4\pi$.

The internal angles of a polygon with n sides add up to $(n - 2) \times 180°$. Therefore, the internal angles of a heptagon add up to $5 \times 180°$.

The total of the angles at a point is 360°. Therefore the total of the angles at the seven vertices of the heptagon is $7 \times 360°$. Since the internal angles have total $5 \times 180°$, the total of the external angles is $7 \times 360° - 5 \times 180° = 7 \times 360° - \frac{5}{2} \times 360° = \frac{9}{2} \times 360°$.

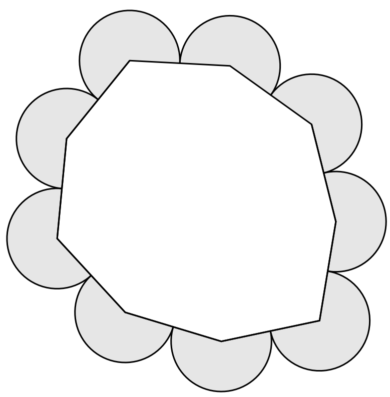
Therefore the total shaded area is $\frac{9}{2} \times 4\pi$, that is, 18π.

For investigation

20.1 Explain why the internal angles of a polygon with n sides add up to $(n - 2) \times 360°$.

20.2 The above solution uses the fact that, if the total of the angles subtended by sectors of circles with the same radius is $x \times 360°$, then the total area of the sectors is $x \times$ the area of one circle with the same radius. Explain why this is true.

20.3 What would the total shaded area be in the similar problem in which the heptagon is replaced by an nonagon with sides all of length 2?



20.4 Let n be a positive integer with $n \geq 3$. Find the formula, in terms of n, for the total shaded area in the similar problem in which the heptagon is replaced by a polygon with n sides each with length 2.

21. *Brachycephalus* frogs are tiny – less than 1 cm long – and have three toes on each foot and two fingers on each 'hand', whereas the common frog has five toes on each foot and four fingers on each 'hand'.

Some *Brachycephalus* and common frogs are in a bucket. Each frog has all its fingers and toes. Between them they have 122 toes and 92 fingers.

How many frogs are in the bucket?

A 15 B 17 C 19 D 21 E 23

SOLUTION **A**

Let b be the number of *Brachycephalus* frogs in the bucket and let c be the number of common frogs in the bucket.

A *Brachycephalus* frog has three toes on each foot and two fingers on each 'hand'. Therefore, in total, it has 6 toes and 4 fingers. A common frog has, in total, 10 toes and 8 fingers.

Therefore, between them b *Brachycephalus* frogs and c common frogs have $6b + 10c$ toes and $4b + 8c$ fingers. Hence, from the information given in the question, we have

$$6b + 10c = 122,$$

and

$$4b + 8c = 92.$$

Subtracting the second equation from the first, we obtain

$$2b + 2c = 30.$$

Dividing the last equation by 2, we deduce that

$$b + c = 15.$$

Therefore, the total number of frogs in the bucket is 15.

FOR INVESTIGATION

21.1 Note that in the above solution we calculated the total number of frogs without working out the number of frogs of each of the two species. How many *Brachycephalus* frogs are in the bucket, and how many common frogs?

21.2 How many frogs would there have been in the bucket if between them they had 62 toes and 44 fingers?

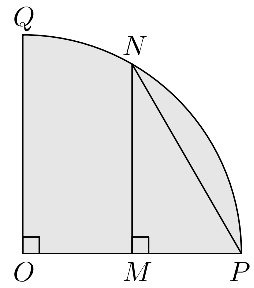
21.3 How many frogs would there have been in the bucket if between them they had t toes and f fingers?

21.4 Assume that a bucket contains an assortment of *Brachycephalus* and common frogs, and that each frog has all its fingers and all its toes. What are the possible values for the number t of toes and the number f of fingers that they have between them?

22. The diagram shows an arc PQ of a circle with centre O and radius 8. Angle QOP is a right angle, the point M is the midpoint of OP and N lies on the arc PQ so that MN is perpendicular to OP.

Which of the following is closest to the length of the perimeter of triangle PNM ?

A 17 B 18 C 19 D 20 E 21

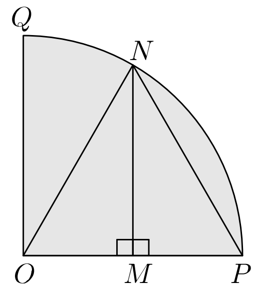


Solution **C**

Because NM is perpendicular to OP, the triangle OMN has a right angle at M. Therefore, by Pythagoras' Theorem,

$$OM^2 + NM^2 = ON^2.$$

Since M is the midpoint of OP and the circle has radius 8, it follows that $OM = 4$. Also $ON = 8$. Therefore, by the above equation $NM^2 = 8^2 - 4^2 = 64 - 16 = 48$. It follows that $NM = \sqrt{48}$.



In the right-angled triangle NMP, we have $NM^2 = 48$ and $MP^2 = 4^2 = 16$. Therefore, by Pythagoras' Theorem applied to this triangle, $NP^2 = MN^2 + MP^2 = 48 + 16 = 64$, and therefore $NP = 8$.

It follows that the perimeter of the triangle PNM is $MP + PN + NM = 4 + 8 + \sqrt{48} = 12 + \sqrt{48}$. Because $\sqrt{48}$ is close to 7, the perimeter is close to $12 + 7 = 19$.

For investigation

22.1 In the argument above we used Pythagoras' Theorem to show that $PN = 8$. It is also possible to prove that $PN = ON = 8$, by showing that the triangles OMN and PMN are congruent. Explain how this may be proved.

22.2 The argument above relied on the rather vague statement that because $\sqrt{48}$ is close to 7, it follows that $12 + \sqrt{48}$ is close to 19. To give a more precise argument that, of the given options, $12 + \sqrt{48}$ is closest to 19, we need to show that $18.5 < 12 + \sqrt{48} < 19.5$. These inequalities are equivalent to $6.5 < \sqrt{48} < 7.5$. Verify that these inequalities are correct.

23. Two brothers and three sisters form a single line for a photograph. The two boys refuse to stand next to each other.

How many different line-ups are possible?

A 24 B 36 C 60 D 72 E 120

Solution **D**

We first count the number of ways in which two brothers and three sisters can stand in line without two brothers being next to each other, without regard to which particular brother and which particular sister is in any given position.

Counting from the left, the first brother can be in the first, second or third position. (He cannot be in fourth position as then the second brother from the left would be next to him in fifth position.)

If the first brother is in first position, the second brother can be in third, fourth or fifth position. If the first brother is in second position, the second brother can be in fourth or fifth position. If the first brother is in third position, the second brother can only be in fifth position.

It follows that there are six ways to arrange where the brothers and the sisters are positioned. These are listed below using B to represent a brother and S to represent a sister.

$$B, S, B, S, S$$
$$B, S, S, B, S$$
$$B, S, S, S, B$$
$$S, B, S, B, S$$
$$S, B, S, S, B$$
$$S, S, B, S, B$$

For any given two positions which the two brothers occupy, they can be put in these positions in 2 ways. If the brothers are $B1$ and $B2$, these are $B1, B2$ and $B2, B1$.

For any given three positions which the three sisters occupy, they can be put in these positions in 6 ways. If the sisters are $S1$, $S2$ and $S3$, these are $S1, S2, S3$; $S1, S3, S2$; $S2, S1, S3$; $S2, S3, S1$; $S3, S1, S2$ and $S3, S2, S1$.

Since the 6 choices of which positions the brothers occupy, the 2 choices of how the brothers are put in these positions, and the 6 choices of how the sisters are put in the remaining positions may be made independently, the total number of different line-ups is given by $6 \times 2 \times 6 = 72$.

For investigation

23.1 How many different line-ups are possible in the similar problem with two brothers and four sisters?

23.2 Let n be a positive integer. Find the formula, in terms of n, for the number of line-ups that are possible in the similar problem with two brothers and n sisters.

24. The nth term of a certain sequence is calculated by multiplying together all the numbers $\sqrt{1+\frac{1}{k}}$, where k takes all the integer values from 2 to $n+1$ inclusive. For example, the third term in the sequence is $\sqrt{1+\frac{1}{2}} \times \sqrt{1+\frac{1}{3}} \times \sqrt{1+\frac{1}{4}}$.

What is the smallest value of n for which for the nth term of the sequence is an integer?

A 3 B 5 C 6 D 7 E more than 7

Solution **C**

Let n be a positive integer. The nth term of the sequence is

$$\sqrt{1+\frac{1}{2}} \times \sqrt{1+\frac{1}{3}} \times \sqrt{1+\frac{1}{4}} \times \cdots \times \sqrt{1+\frac{1}{n+1}}.$$

This expression may be rewritten as

$$\sqrt{\frac{3}{2}} \times \sqrt{\frac{4}{3}} \times \sqrt{\frac{5}{4}} \times \cdots \times \sqrt{\frac{n+2}{n+1}},$$

which is equivalent to

$$\sqrt{\frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \cdots \times \frac{n+2}{n+1}}.$$

In the product above we may cancel all the terms other than the denominator of $\frac{3}{2}$ and the numerator of $\frac{n+2}{n+1}$. In this way the above expression may be simplified to

$$\sqrt{\frac{n+2}{2}}.$$

For this to be an integer, we require that $\frac{n+2}{2}$ be a square and hence that $n+2$ is twice a square. Because $n+2>2$, the least possible value of $n+2$ is 2×2^2, that is, 8. Now $n+2=8$ for $n=6$.

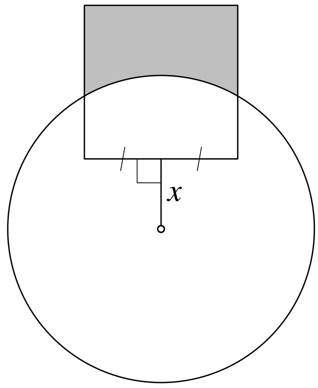
Therefore 6 is the smallest value of n for which the nth term of the sequence is an integer.

For investigation

24.1 Which is the least value of n, such that $n>6$ and the nth term of the sequence is an integer?

24.2 Show that there are infinitely many different positive integers n for which $\sqrt{\frac{n+2}{2}}$ is an integer.

25. The diagram shows a circle with radius 2 and a square with sides of length 2. The centre of the circle lies on the perpendicular bisector of a side of the square, at a distance x from the side, as shown. The shaded region—inside the square but outside the circle—has area 2.



What is the value of x?

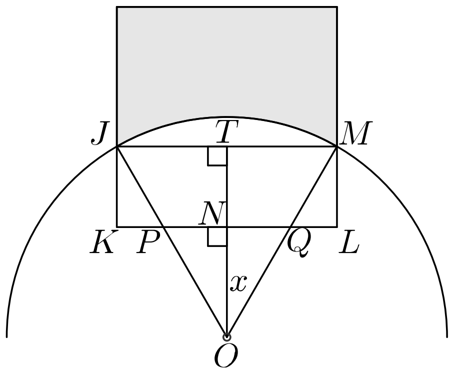
A $\frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1$ B $\frac{\pi}{3} + \frac{\sqrt{3}}{4} - 1$

C $\frac{\pi}{3} + \frac{1}{2}$ D $\frac{\pi}{3} + 1$

E $\frac{\pi}{3}$

SOLUTION **A**

We let O be the centre of the circle, and the points J, K, L, M, N, P, Q and T be as shown in the figure.



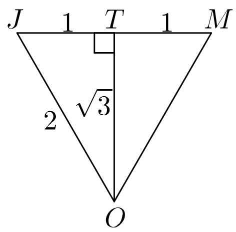
The area that is shaded is the area of the square less the area of the part of the square that is inside the circle. Therefore we begin by working out this latter area.

The part of the square that is inside the circle is made up of the segment of the circle cut off by the chord JM and the rectangle $JKLM$. We let the areas of these regions be A_S and A_R, respectively.

From the figure above we see that A_S equals the area of the sector of the circle subtended by the arc JM at the centre O of the circle, less the area of the triangle JOM.

In the triangle JOM, JO and MO have length 2, because they are radii of the circle, and JM has length 2, because it has the same length as the sides of the square. It follows that the triangle JOM is equilateral. Hence $\angle JOM = 60°$. Because $60°$ is one-sixth of a complete revolution, the area of the sector of the circle subtended by the arc JM is one-sixth of the area of the circle of radius 2. Hence the area of this sector is $\frac{1}{6} \times \pi(2^2) = \dfrac{2\pi}{3}$.

We now work out the area of the equilateral triangle JOM which has side length 2.



We note that T is the point on JM such that OT is perpendicular to JM. Hence the triangles JOT and MOT are congruent. Therefore $JT = TM$, and hence JT has length 1. By Pythagoras' Theorem applied to the triangle JTO, We have $OT^2 = JO^2 - JT^2 = 2^2 - 1^2 = 3$. Therefore OT has length $\sqrt{3}$.

Therefore the triangle JOM has a base of length 2 and height $\sqrt{3}$. Hence the area of this triangle is $\frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$.

It follows that $A_S = \dfrac{2\pi}{3} - \sqrt{3}$.

We now work out the value of A_R which is the area of the rectangle $JKLM$.

We have already noted that JM has length 2. The length of KJ is the same as the length of NT. This length is $OT - ON = \sqrt{3} - x$. So $KJ = \sqrt{3} - x$. Therefore

$$\begin{aligned} A_R &= JM \times JK \\ &= 2(\sqrt{3} - x). \end{aligned}$$

Therefore the area of the square that is inside the circle is

$$\begin{aligned} A_S + A_R &= \left(\frac{2\pi}{3} - \sqrt{3}\right) + 2(\sqrt{3} - x) \\ &= \frac{2\pi}{3} + \sqrt{3} - 2x. \end{aligned}$$

The area of the square is 4. Therefore for the area outside the circle to be 2, we require that the area inside the circle is also 2. That is, we require that

$$2 = \frac{2\pi}{3} + \sqrt{3} - 2x.$$

This last equation may be rearranged as

$$2x = \frac{2\pi}{3} + \sqrt{3} - 2,$$

from which it follows that

$$x = \frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1.$$

For investigation

25.1 Prove the following statements that are used in the above solution.

(a) $JKLM$ is a rectangle.

(b) JM has length 2.

(c) The triangles JOT and MOT are congruent.

(d) $\angle NPO = \angle JPK = 60^\circ$.

(e) N lies on the line OT.

(f) $NT = KJ$.